A Note on the Approximate Reconstruction of a Function and its Derivatives from Noisy Data

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Introduction

Given an open bounded interval Ω in $\Re$ and a function $u(x)$ with generalized derivatives in $L_2(\Omega)$, we show how to construct a sequence of smooth functions that approximates u and its derivatives in Ω, up to the boundaries, from noisy data.

Keywords

Ill-Posed Problems, Regularization, Mollification, Differentiation of Noisy Data.

Main Results

The main results are summarized in the following theorem.

Theorem: Let Ω be a bounded domain in $\Re$ and let u and u^{ε} denote functions which, together with its generalized derivatives $D^{\alpha}u$, $\alpha \leq m+1$, are in $L_2(\Omega)$. If $\| u - u^{\varepsilon} \|_{L_2(\Omega)} \leq \varepsilon$, then there exists a mollified function $(u^{\varepsilon})_{\delta} \in C^{\infty}(\overline{\Omega})$, with radius of mollification δ, satisfying

$$\| D^{\alpha}u - D^{\alpha}(u^{\varepsilon})_{\delta} \|_{L_2(\Omega)} \leq C_{\alpha+1}\delta + C_{\alpha,|\Omega|}\frac{\varepsilon}{\delta^{\alpha+1/2}} .$$

Moreover, there exists a sequence of mollifiers $\{(u_k^{\varepsilon})_{\delta}\}$, in $C^{\infty}(\overline{\Omega})$, such that

$$\lim_{\delta \to 0, k \to \infty} \| D^{\alpha}u - D^{\alpha}(u_k^{\varepsilon})_{\delta} \|_{L_2(\Omega)} = 0, \; \alpha \leq m+1.$$

Remarks:

1. In the absence of noise in the data, the first term in the last formula shows that the approximation by mollification is consistent with the operation of differentiation.
2. In the presence of noise in the data, the second term shows that the approximation by mollification is stable.
3. The estimates are specific to mollification with a family of Gaussian kernels.
4. The numerical strategy consists on using the sequence of mollifiers to attempt to approximate the mollified derivatives of the noisy data function.

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References

[1] Schwartz, L., *Theorie des Distributions*, Vol. 1, Hermann, Paris, 1950.
[2] Yosida, K., *Functional Analysis*, 3rd ed., Springer-Verlag, New York, 1971.
[3] Murio, D.A., *Mollification and Space Marching*, Chapter 4 in Inverse Engineering Handbook, K. Woodbury, ed., CRC Press, Boca Raton, Florida, 2002.